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A Spatial Extension to Growth Models

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Abstract

This paper presents an introduction into spatial extensions to growth models, in particular to the Solow Model. An overview over the two most prominent approaches, discrete space modelling and continuous space modelling, is followed by a detailed account of the continuous space case for the Solow model. The model introduces inter-spatial capital dynamics, complementing the regular intertemporal dynamics. A steady state solution is found for the regular case. An extension is introduced for the case with exogenous growth induced by technological advance and population dynamics, resulting in a balanced growth path. The spatial extension to the Solow model provides a successful merger of the two previously seperate fields of new economic geography and economic growth theory, opening new perspectives to explain economic phenomena in the interplay of space and time.

Keywords: Solow model, spatial modelling, continuous space, discrete space

Contents

1 Introduction

The rediscovery of space as a major feature in economic analysis has gained significance in the last two decades, with important contributions from Krugman (1991), Krugman & Venables (1995) and Fujita *et al.* (1999). Giving new explanations to phenomena of agglomeration, migration and other space-related economic activities, research in this field has gained popularity.

At the same time, new economic geography did not blend in with growth theory in the first years of its emergence. Brito (2004b) calls it "a puzzle" that a stylized fact as convergence among countries is studied intensively by growth theory[1], while at the same time new economic geography is explaining agglomeration and dispersion forces with a focus on international economics.

As possible causes for this division between the two fields Camacho & Zou (2004) suggest the two main characteristics of contributions in new economic growth theory. A discrete space structure, where traditionally predominantly only two locations in space are considered [2], and the absence of capital accumulation, which removes the basis for any contribution from economic growth theory. As to the first proposition, Ertur & Koch (2007) show that discrete spatial modeling is not necessarily an obstacle to implementing spatial features into growth models.[3] The second proposition however is not deniable, and Brito (2004a) and Camacho & Zou (2004) set out to change this. Their common approach, by combining the spatial ingredients of new economic geography with its agglomeration and dispersion forces, together with traditional growth models of Solow (1956) and Ramsey (1928) opens the door to research of interaction between growth in time and space.

[1]see e.g. Barro & Sala-i-Martin (2003).

[2]see and Fujita *et al.* (1999).

[3]see subsection 2.1 for an overview over the concept of discrete space in growth theory.

The aim of this paper is to provide a general overview over approaches to spatially augmented growth models, in particular in the case of the Solow model. The research in this field, spearheaded by Camacho & Zou (2004), Brito (2004a) and Ertur & Koch (2007), using different techniques of modeling spatial components in growth models, has longed to combine the achievements of new economic geography with economic growth theory.

The structure of the paper is as follows: Section 2 gives an overview over the two predominant, yet relatively new approaches to extending the original Solow model with spatial features. The discrete space approach pioneered by Ertur & Koch (2007) will be portrayed in subsection 2.1. Continuous space, a concept introduced to the Solow model by Camacho & Zou (2004) and Brito (2004b) and applied to other prominent growth models, will receive a first glance in subsection 2.2. Here the differences between the two approaches will be highlighted. An in-depth analysis of assumptions and results of the continuous space concept will then be presented in section 3, primarily refering to and synthesizing Camacho & Zou (2004), Brito (2004a) and Brito (2004b). The model will be presented from scratch, starting with the basic assumptions with regards to the household in time and space in subsection 3.1, followed by the description of the particularly important characteristics of space in subsection 3.2. A discussion of the choice of boundary conditions in subsection 3.3 is followed by the articulation of the problem and its steady state solution in subsection 3.4. Subsection 3.5 offers a glance at first further extensions by Brito (2004b) resulting in unbounded growth. Section 4 concludes.

2 Spatial Solow Models

There are two general directions in the field of spatial extensions to growth models, differing in the way of both modelling space technically and the transmission channels for growth effects. These two streams of research either express space in a continuous form, in the form of a finite or infinite number of locations "‘next to eachother"’; or space is expressed in a discrete way, with a finite number of household and their interdependence modelled with matrices. The continuous spatial approach, as will be seen in section 3, puts an emphasys on the *location* and the interaction and interdependence between households at different locations occurs via exchange of capital flows. On the otherhand, the discrete spatial approach, which receives an introduction into its main philosophy in subsection 2.1, does not focus on the location of households in space. The importance in discrete space lies in the *connection* between households, no matter where located. Additionally, the interaction and interdependence of households comes from technological spillovers due to spatial friction.

Each of the two approaches hence brings about own particularities in technical terms, as well as in their respective results. These distinctions will be described in the following. The discrete spatial modelling approach will be highlighted and its general philosophy portrayed. The continuous approach will receive a deeper analysis, starting with a short introduction in subsection 2.2 and then a detailed view in section 3.

2.1 Discrete Spatial Modelling

The discrete space setting assumes a number of distinct households, which produce and consume independently, yet are spatially interdependent with other neighboring or connected households.

Ertur & Koch (2007) set up a framework in which spatial interdependence between countries is a crucial factor for growth resulting from knowledge spillovers. Following Romer (1986), it is assumed that capital investment also increases level of technology. Fundamental to the idea of this discrete setup is the idea of connections between households. Ertur & Koch (2007) build the framework upon the following two equations, the rest of the concept is to a high degree analogous to the original model by Solow (1956):

$$Y_i(t) = A_i(t) K_i^{\alpha}(t) L_i^{1-\alpha}(t)$$

where the technology factor A_i is defined as

$$A_i(t) = \Omega(t) k_i^{\sigma}(t) \prod_{j \neq i}^{N} A_j^{\gamma w_{ij}}(t) \tag{1}$$

For a number of N distinct households, each household is characterized by its own production function Y_i, and the respective stock of capital K_i and labor L_i.

The technology factor A_i is paramount to the idea of connectivity to the model. The individual factor A_i is composed of several parts, as can be seen in equation (1). There is an exogenous component $\Omega(t) = \Omega(0)e^{\mu t}$, which is, as in a regular textbook Solow model the technological progress, with μ as the constant rate of growth of technology. The second part in the technology function of the setup in Ertur & Koch (2007) is the inclusion of capital per worker k_i, therefore including a spillover treatment of knowledge through capital investment in the sense of Romer (1986). The third and last part in equation (1) models the interdependence of households through technological spillover effects from neighboring households. The spatial externalites originate from the geometric average of technology factor of connected households $\prod_{j \neq i}^{N} A_j^{\gamma w_{ij}}(t)$. The variable w_{ij} shows the strenght of a connection or spatial friction, as Ertur & Koch (2007) put it. The respective values of w_{ij} show the degree of connectivity between households i

and j. Junior (2010), in an application of the framework to estimate the effects of connectedness between different Brazilian regions, uses the relative border length between the respective states. The productivity level of a household is therefore in total affected by the exogenous and homogeneous level of productivity, the amount of capital per capita in the respective household, and the inherited knowledge spillovers from nearby household technologies.
Aggregating the individual households' technology function yields the global technology function, showing the interdependence nicely in a matrix form:

$$\mathbf{A} = \mathbf{\Omega} + \phi k + \gamma \mathbf{W}\mathbf{A},$$

where $\mathbf{A}$,$\mathbf{\Omega}$ and $\mathbf{k}$ are the vectors of logarithms of levels of technology, the shared technological "'basis"' and the level of physical capital per capita respectively. $\mathbf{W}$ is the Markov matrix with values describing the connectivity between households.

The general idea in this discrete space setup of a Solow model by Ertur & Koch (2007) hence is to model technological spillover effects and the impact on growth of neighboring households. The outcome of the model with regard to a steady state is similar to the original text book Solow model, with the difference of a spatial multiplier effect. For a number of households $N = 1$, one household in complete autarky, the result is therefore equivalent to the one from the non-spatial version of the model. The higher the spatial-interdependence, the higher the experienced spillover effects for the model with $N > 1$.

The output of a household, Ertur & Koch (2007) show, positively depends on the "home" savings rate and negatively on its population growth rate. The difference occurs in the impact of "foreign" savings rate, which is also positive, and "foreign" population growth rate, which is negative. The transmission channel for this spatial multiplier effect is the described technological transfer through connectedness.

2.2 Continuous Spatial Modelling

In a continuous space setting, spatial interdependence is modelled by introducing space as a seperate dimension next to time. Camacho & Zou (2004) and Brito (2004a) develop a framework in which capital does not only rely on savings, but also allows for flows of capital to different locations.

There exist two major differences between the discrete space approach and the continuous space approach: the transmission channel that causes the impact on growth through spatial interdependence and interaction; and the technical way of modelling this effect of inclusion of space. While in the discrete framework the mere *connections* of a household or between households have an impact on the production function through technological spillovers, in the continuous framework the *location* of a household plays the crucial role.

The fundamental idea of the Solow model with continuous space is the introduction of the additional dimension of space, next to time, into the original model by Solow (1956). The transmission channel of the effect of interdependence lies in the inter-spatial capital flows, in addition to the usual inter-temporal dynamics of the classic model. This idea, first introduced by Brito (2004a) and then refined by Camacho & Zou (2004) and Brito (2004b) for use with the Solow model, lay the groundwork for subsequent further explorations in the frontier of space, as a spatial dimension to the variables could also be introduced into other prominent growth models.

Closely related to the research on the spatial Solow model has been work on the Ak model, more or less a simplified version of the same model with non-diminishing returns of capital. Being utilized in Camacho & Zou (2004) to exemplify the dynamics of the framework, it also receives attention in Boucekkine *et al.* (2010) with a particular modelling of space in the form of the unit circle.[4] Introducing the framework of continuous space to the Ramsey model, Brito (2004a) and then later Boucekkine *et al.* (2009) again employ the same basic assumptions of space and dynamics of capital flows between locations to its original text book version .

In the following section 3, the concept of continuous space in the Solow model will be explained in detail.

[4]see here also subsection 3.3.

3 The Solow Model in Continuous Time and Space

The most comprehensive framework for a Solow model in continuous space has been set up by Camacho & Zou (2004) and Brito (2004b). Unlike in a regular textbook Solow model, each variable is not only characterized by a point in time t , but also a point in space x.

Introducing space as an additional dimension to time has some decisive consequences on the results of the model. At first, it changes the fundamental decision of investment in capital from a purely time-dependent choice, to a time- and space-dependent one. The original model allows so save some of the income of a period to be invested into capital which is available in the following period. The continuous space version enriches this choice by the option to invest in other locations than "home".

The space and time dependence is shown by $(x, t) \in \mathbb{R} \times \mathbb{R}_+$. Space is continuous, infinite and unidimensional, while time is continuous and positive. Capital hence flows in both dimensions.

3.1 The household in time and space

The model is set up with the usual components of a Solow model. $K = K(x, t)$ is the capital stock at time t and location x, $L = L(x, t)$ the respective labor input. Productivity, savings and technological progress are assumed heterogeneous with respect to location, yet constant over time. The household, produces $Y = Y(x, t)$, and then consumes $C = C(x, t)$ and invests $I = I(x, t)$. It is located on the real line.[5]

The model assumes a regular neoclassical production,

$$Y(x, t) = F\left[A(x), K(x, t), L(x, t)\right] \quad (2)$$

[5]This assumption brings some important implications, which will be discussed in 3.3.

As specified in Camacho & Zou (2004), the function $F(\cdot)$ is satisfies the following assumptions:[6]

- $F(\cdot)$ is non-negative, increasing and concave;
- $F(\cdot)$ satisfies the Inada conditions:
  1. $f(0) = 0$
  2. $\lim_{k\to\ 0} f'(k) = +\infty$
  3. $\lim_{k\to+\infty} f'(k) = 0$

The produced good can be either consumed or saved and reinvested, as the produced good is homogeneous for consumption and capital investment. The household's consumption is defined as the residual from the individual savings, therefore:

$$C(x,t) = (1 - s(x))Y(x,t)$$

with the savings rate $0 < s(x) < 1$ and $x \in \mathbb{R}$.
Accounting for the possibility of investing in other locations than home, the investment of the household consists of the savings and the current account, the net trade balance:

$$I(x,t) = s(x)Y(x,t) + \tau(x,t) \tag{3}$$

The net trade balance $\tau(x,t)$ is defined in the usual terms as the difference between exports, hence capital outflows, and import, hence capital inflows.

The investment $I(x,t)$ is also reflected in the change in capital stock, as in investment has an instantaneous effect. The total investment, coming from local

[6]Camacho & Zou (2004) use a labor intensive Cobb-Douglas production function, which enables to make dynamic simulations, which are showcased in subsection 3.4. This paper follows Brito (2004a) and Brito (2004b) with a nonspecified neoclassical production function, allowing to easier model labor mobility in subsection 3.5.

or foreign investment, is used to build up new capital and make up for the loss in stock of capital due to depreciation:

$$I(x,t) = \frac{\partial K(x,t)}{\partial t} + \delta(x)K(x,t). \quad (4)$$

$\delta(x)$ is the depreciation rate, which is assumed to be heterogeneous with respect to location, but constant over time. Setting equal equations 3 and 4, both the demand and supply side of capital investment become visible:

$$\frac{\partial K(x,t)}{\partial t} + \delta(x)K(x,t) = s(x)Y(x,t) + \tau(x,t). \quad (5)$$

3.2 Regions

Fundamental to the concept of the introduction of space into the Solow model is the modelling of inter-spatial capital dynamics, as opposed to the purely inter-temporal capital flows in the original version without inclusion of space.

Taking the economy as a whole, all investments must originate from some location x_0 and then flow to another location x_1 with $x_0, x_1 \in \mathbb{R}$. Hence, the whole economy with all locations $x \in \mathbb{R}$, can be seen as a closed economy. The cumulative differences between investment and savings, equaling the current account, has to be balanced and zero. Integrating over all locations equation (3) yields:

$$\int_{\mathbb{R}} S(x,t) - I(x,t)dx = \int_{\mathbb{R}} \tau(x,t)dx = 0$$

The same relationship also holds for any closed region $[a,b]$ within $\mathbb{R}$, so that

$$\int_{[a,b]} S(x,t) - I(x,t)dx = \int_{[a,b]} \tau(x,t)dx = 0$$

$$\Leftrightarrow \int_{[a,b]} S(x,t) - I(x,t) - \tau(x,t)dx = 0$$

This relationship is equal to zero, when the region is completely closed to trade, or in autarky. $\tau(x,t) = 0$ could however also imply an open region with a balanced current account.

An open region is characterized by flows in and out of its boundaries, allocating capital from households with low marginal productivity of capital to ones with high marginal productivity outside of the region. $\tau(x,t) \neq 0$ is a sufficient, but not necessary condition for an open region:

$$\int_{[a,b]} \tau(x,t)dx \neq 0$$

A $\tau(x,t) \neq 0$ implies a net flow of capital to or from location x, showing inter-spatial capital dynamics. If $\tau(x,t) > 0$ the household is a net exporter of capital, for $\tau(x,t) < 0$ the household is a net importer, attracting capital from outside.

Following Isard & Liossatos (1979) and Camacho & Zou (2004), the flow of capital is equal to the gradient of capital distribution measured at the borders of the region $[a,b]$.[7] The gradient, as the direction of the flow, can figuratively be the difference of capital flows at both borders a and b. The relation hence:

$$\int_{[a,b]} \tau(x,t)dx = -\left[\frac{\partial K(b,t)}{\partial x} - \frac{\partial K(a,t)}{\partial x}\right] \tag{6}$$

This relationship shows the direction of capital flows from rich regions to poor regions. Rich regions export, as capital experiences diminishing returns, to poorer regions, where marginal returns are higher.

Following Camacho & Zou (2004) and applying the fundamental theorem of calculus, this relationship can be rewritten as:

$$\begin{aligned}\left[\frac{\partial K(b,t)}{\partial x} - \frac{\partial K(a,t)}{\partial x}\right] &= \int_{[a,b]} \frac{d}{dx}\left(\frac{\partial K(x,t)}{\partial x}\right)dx \\ &= \int_{[a,b]} \frac{\partial^2 K(x,t)}{\partial x^2}dx \end{aligned} \tag{7}$$

[7]As Brito (2004a) notes, Isard & Liossatos (1979) cover this property extensively.

Inserting (7) into (6), this yields:

$$\int_{[a,b]} \tau(x,t)dx = - = \int_{[a,b]} \frac{\partial^2 K(x,t)}{\partial x^2}dx \tag{8}$$

In Solow (1956), capital accumulates described by the following function for a household in autarky:

$$\frac{\partial K(x,t)}{\partial t} = s(x)F(\cdot) - \delta K(x,t)$$

For $|a - b| \to 0$ and following Camacho & Zou (2004) in applying the Hahn-Banach theorem to equation (8), the respective function for an open household therefore follows from equation (5):

$$\frac{\partial K(x,t)}{\partial t} = s(x)F(\cdot) - \delta K(x,t) + \frac{\partial^2 K(x,t)}{\partial x^2}$$

$$\frac{\partial K(x,t)}{\partial t} - \frac{\partial^2 K(x,t)}{\partial x^2} = s(x)F(\cdot) - \delta K(x,t) \tag{9}$$

Comparing this capital accumulation function with the non-spatial version, the interspatial dynamics of capital become clear. Capital flows both over time via savings and over space via transfer of capital from one location to another.

Assuming that there is no barrier to capital flow, or, as Camacho & Zou (2004) suggest no adjustment speed, capital flow will elimininate geographic differences in capital stock. Abundance implies a lower marginal productivity of capital through diminishing returns, implied by the Inada conditions,[8] and will therefore not be attractive to investment. Scarcity on the other hand will allow for higher marginal returns to capital, therefore attracting capital flows towards this location, leading to capital movements eliminating differences of capital stock through ”trade”, or better ”foreign investment”. These dynamics are captured by the heat equation (9).

[8]see beginning of section 3.1.

3.3 Boundaries

As Boucekkine *et al.* (2010) note, the choice of boundary conditions for this problem of partial differential equations is essential. The solution to the problem is extremely sensitive to this choice.

Camacho & Zou (2004) choose the following boundary condition for the distribution of capital in space:

$$\lim_{x \to +-\infty} \frac{\partial K(x,t)}{\partial x} = 0 \tag{10}$$

The choice of this boundary condition implies that far away from center there is no capital flow. Brito (2004a) however specify the boundary condition with:

$$\lim_{x \to +-\infty} \frac{\partial K(x,t)}{x} = 0$$

This choice implies not only no flow at the tails of the distribution of capital stock, but also no capital at all. The tails of the distribution are hence bounded functions. Brito (2004a) gives the advantages and disadvantages of applying this exact condition, which is weaker than a regular Neumann boundary, chosen by Camacho & Zou (2004).[9]

This paper follows the choice of Camacho & Zou (2004) given in equation (10) as the boundary condition, to allow for a solution to the partial differential equation.

Boucekkine *et al.* (2010) avoid the arbitrary selection of a boundary condition

[9] see pp. 14 - 15 in Brito (2004a) for a discussion. Boucekkine *et al.* (2009) give a justification for the Ramsey case, choosing the Neumann boundary condition like Camacho & Zou (2004). Boucekkine *et al.* (2010) gives further cases and respective justifications, e.g. for a compact interval on the real line.

by rendering space in a unit circle. With $x \in \mathbb{R}^2 : |x| = 1$, the specification of boundary conditions for space becomes obsolete.

3.4 Problem and Steady State Solution

Given an initial distribution of capital $K(x, 0)$, the problem can then be written as:

$$P = \begin{cases} \frac{\partial K(x,t)}{\partial x} - \frac{\partial^2 K(x,t)}{\partial x^2} - s(x)F(\cdot) + \delta K(x,t) = 0 \\ K(x,0) > 0, x \in \mathbb{R} \\ \lim_{x \to +-\infty} \frac{\partial K(x,t)}{\partial x} = 0 \end{cases}$$

As in the Solow (1956) version without spatial features, the steady state solution is characterized by

$$\frac{\partial K(x,t)}{\partial t} = 0.$$

The stock of capital does not change over time anymore, but can still vary over space. As technology $A(x)$ is only location-dependent and not time-dependent, implying no technological progress along the time horizon, the problem is hence:

$$P_S = \begin{cases} \frac{\partial^2 K(x,t)}{\partial x^2} + s(x)F(\cdot) - \delta K(x,t) = 0 \\ \lim_{x \to +-\infty} \frac{\partial K(x,t)}{\partial x} = 0 \end{cases}$$

which, following Camacho & Zou (2004), can be solved by reducing this partial differential equation into an ordinary differential equation:

$$\frac{\partial K(x,t)}{\partial x} = w(x)$$

and therefore

$$\frac{\partial w(x)}{\partial x} = \frac{\partial^2 K(x,t)}{\partial x^2}.$$

The steady state solution is then given by

$$K(x) = \int_{-\infty}^{x} w(z)dz$$

and

$$w(x) = \int_{-\infty}^{x} \left(-s(z)F(\cdot) + \delta K(z)\right) dz.$$

Any solution must also satisfy the boundary condition described in subsection 3.3, so in this case:

$$\lim_{x \to +-\infty} \frac{\partial K(x)}{\partial x} = 0$$

and

$$\lim_{x \to +-\infty} w(x) = 0$$

respectively. If then the stock of capital K is constant at every location x, so that

$$s(x)F(\cdot) = \delta K(x), \vee x \tag{11}$$

K is a particular solution to P_S.[10]

Camacho & Zou (2004) illustrate the results with several different sets of visualizations of dynamic simulations. They examine three cases, each modelling a different distribution of initial values for capital and technology:

1. homogeneous distribution of both initial capital and technology: as all households along the real line are considered to have homogeneous endowments at their disposal and are provided with the same level of productivity, investors are indifferent towards choice of location. All households therefore experience growth at the same rate.

[10]As Camacho & Zou (2004) note, these stationary solutions to the problem are not unique though.

2. homogeneous distribution of technology, but heterogeneous distribution of initial capital: the expected result of a wealth difference-smoothing dynamic that transfers capital from low marginal return, hence rich areas, to high marginal return, yet poor areas, leads to a sigma-convergence. In the long-run therefore, as expected, initial differences are overcome and growth solely depends on the level of productivity.

3. heterogeneous distribution of technology with or without additional heterogeneity in initial capital distribution: as a transfer of technology does not happen in the model,[11] there is no sigma-convergence. Even over the long-run, differences persist, as the smoothing dynamics of capital flows do not make up for the difference in level of productivity. The households, located along the real line, will experience different levels of wealth according to the respective technology at their disposal.[12]

The last case above describes the result of a variation in level of productivity over space. The following section 3.5 will give a brief introduction into the extension by Brito (2004b), incorporating a variation of technology over time, namely technological progress.

3.5 Technological Progress and Labor Mobility: Unbounded Growth

So far the model has only reproduced the standard Solow model, delivering similar results to Solow (1956), yet in the spatial context. The steady state solution however is not taking into account any technological progress. Brito (2004b) extends the basic framework previously established by Camacho & Zou (2004) with

[11] as opposed to the discrete setup by Ertur & Koch (2007), briefly introduced in section 2.

[12] For the illustrations with respective values see Camacho & Zou (2004) pp. 7 - 9.

technological advance and labor mobility. The inclusion of these new parameters allows for additional dynamics, resulting in a balanced growth path and hence unbounded growth rather than a steady state.
Technological progress is modelled easily by enhancing the productivity parameter in the production function to be sensitive to time. In the previous setting it was only location-dependent. The technical progress is assumed to be labor augmenting, hence:

$$A(x, t) = a(x)e^{\gamma t} \tag{12}$$

Technology is improving with time, at a constant growth rate of $\gamma \geq 0$, hence the exponential function with time dependence. Technology can still also vary over space, as seen by the location dependence of $a(x)$. As technological spillovers still do not occur, a onetime difference in technology between locations resembles a permanent difference.

Brito (2004b) additionally introduces labor dynamics, paving the way for another factor to be flowing across space. Labor $L(x, t)$ is now also assumed to grow, at a homogeneous rate of $\mu \geq 0$. Analogous to the level of productivity, the population thus grows exponentially:

$$L(x, t) = l(x)e^{\mu t}, \forall (x, t) \in \mathbb{R} \times \mathbb{R}_+ \tag{13}$$

As this change in population size is homogeneous across all locations, the population size solely depends on the initial distribution of labor at time $t = 0$. Hence mobility for the factor labor is allowed by introducing a migration possibility in the economy. Labor disperses analogous to capital, from regions with low marginal product to regions with high marginal product. Assuming the production function to follow the Inada conditions, as described in the beginning of section 3, regions abundant with labor will experience lower marginal returns to this factor than regions experiencing scarcity, due to diminishing marginal returns to input factors. The change in stock of labor for a point in time and space is described by:

$$\frac{\partial L(x,t)}{\partial t} = \frac{\partial^2 L(x,t)}{\partial x^2} + \mu L(x,t)$$

Brito (2004b) decomposes the changes in the population growth subsequently into an unbounded and a transient component.[13]

The introduction of exogenous growth elements, namely the level of productivity and labor force, results in unbounded growth, as opposed to the bounded growth described by the steady state solution in subsection 3.4. Equivalent to the known results from the Solow model with exogeneous growth, the model experiences growth along a balanced growth path.

Relying on the results from the problem without exogenous growth, the partial differential equation 9 shows the inter-spatial and inter-temporal dynamics for a world composed of open regions. Following Brito (2004b), and inserting equations 13 and 12 into the production function 2 and then into equation 9 yields:

$$\frac{\partial K(x,t)}{\partial t} = \frac{\partial^2 K(x,t)}{\partial x^2} + s(x)F\left(K(x,t), a(x)l(x,t)e^{(\gamma+\mu)t}\right) - \delta K(x,t)$$

Expressing $K(x,t)$ in labor intensive form as $k(x,t)e^{(\gamma+\mu)t}$ then results in:

$$\Leftrightarrow \frac{\partial k(x,t)}{\partial t} = \frac{\partial^2 k(x,t)}{\partial x^2} + s(x)F\left(k(x,t), a(x)l(x,t)\right) - \left(\delta + \gamma + \mu\right)k(x,t) \tag{14}$$

The balanced growth path is defined, as in Barro & Sala-i-Martin (2003), such that:

$$\overline{K}(x,t) = \overline{k}(x,t)e^{(\gamma+\mu)t}, \forall (x,t) \in \mathbb{R} \times \mathbb{R}_+ \tag{15}$$

[13] see Brito (2004b) pp. 5 - 7 for a discussion.

Inserting relation 15 into equation 14 is leading to the solution equivalent to 11 in the steady state case without exogenous growth:

$$\frac{\partial^2 \overline{k}(x)}{\partial x^2} + s(x)F\left(\overline{k}(x), a(x)\overline{l}\right) - (\delta + \gamma + \mu)\,\overline{k}(x) = 0$$
$$\Leftrightarrow \frac{\partial^2 \overline{k}(x)}{\partial x^2} + s(x)F\left(\overline{k}(x), a(x)\overline{l}\right) = (\delta + \gamma + \mu)\,\overline{k}(x)$$

where $\bar{l}$ is a homogeneous distribution of population across all locations.[14]

The result is, as in the case without exogeneous growth, surprisingly simple and elegant.

[14] compare Brito (2004a) p. 6.

4 Conclusion

With the introduction of a spatial extension to growth models the two previously unconnected fields of new economic geography and economic growth theory find together. What Brito (2004b) called a ”puzzle”, the seemingly obvious yet not existing connection between the two research areas, has been solved, with important contributions by Brito (2004a) and Camacho & Zou (2004).

The phenoma, which new economic geography tries to explain, most importantly agglomeration and dispersion forces, are reproduceable with the interspatial dynamics of the proposed framework. At the same time, the absence of capital accumulation, which left one of the main characteristics of many economic observances unaccounted for in the new economic geography, is very present in the framework.

This paper aimed to provide a general overview over the two main approaches to spatial extensions to the Solow model, with a focus on the continuous space modelling pioneered by Brito (2004a) and introduced to the Solow model by Camacho & Zou (2004) and Brito (2004b). The brief introduction of the basic philosophy of Ertur & Koch (2007) makes clear however, that there is, as so often in economic theory, not one best practice, but different approaches give explanations to different phenomena. As described above, the main contribution of the discrete approach to spatial modelling in growth lies in its strength to focus on the *connection* between actors. This could open the door to a utilization of this framework in other related fields, such as economic network theory.

The continuous approach, as shown in detail and derived from scratch synthesizing Brito (2004a), Brito (2004b), Camacho & Zou (2004) and others, focuses on the other hand on the importance of the *location* of the economic agent. The application of this framework is as broad as its discrete sibling, allowing for an implementation in a wide range of other growth models[15] through its very simple strategy of modelling space as an additional dimension next to time.

With the maturity of this merging field between new economic geography and economic growth theory far from being reached, exciting new approaches and advances in the two presented frameworks can be expected. In the world of today, in which interdependence has reached levels prior unknown, and interaction over space being increasingly simple and less costly, spatial features in growth models might become a standard way to explain the very visible dynamics of a global economy.

[15] as research by Boucekkine *et al.* (2009) and Desmet & Rossi-Hansberg (2010) and others shows.

References

Barro, Robert J., & Sala-i-Martin, Xavier. 2003. *Economic Growth, 2nd Edition*. MIT Press Books, vol. 1, no. 0262025531. The MIT Press.

Boucekkine, Raouf, Camacho, Carmen, & Zou, Benteng. 2009. Bridging the gap between growth theory and the new economic geography: the spatial ramsey model. *Macroeconomic Dynamics*, **13**, 20 – 45.

Boucekkine, Raouf, Camacho, Carmen, & Fabbri, Giorgio. 2010. *Spatial dynamics and convergence: The spatial AK model*. Tech. rept. Institut de Recherches conomiques et Sociales de l'Universit catholique de Louvain.

Brito, Paulo. 2004a. The dynamics of growth and distribution in a spatially heterogeneous world. *ISEG Working papers*.

Brito, Paulo. 2004b. A spatial Solow model with unbounded growth.

Camacho, Carmen, & Zou, Benteng. 2004. The spatial Solow model. *Economics Bulletin*, **18**(2), 1 – 11.

Desmet, Klaus, & Rossi-Hansberg, Esteban. 2010. On Spatial Dynamics. *Journal of Regional Science*, **50**(1), 43–63.

Ertur, Cem, & Koch, Wilfried. 2007. Growth, Technological Interdependence and Spatial Externalities: Theory and Evidence. *Journal of Applied Econometrics*, **22**, 1033 – 1062.

Fujita, Masahisa, Krugman, Paul, & Venables, Anthony J. 1999. *The Spatial Economy: Cities, Regions, and International Trade*. MIT Press Books, vol. 1, no. 0262561476. The MIT Press.

Isard, Walter, & Liossatos, Panagis. 1979. *Spatial dynamics and optimal space-time development*. Vol. 4. North-Holland (New York).

Junior, Waldery Rodrigues. 2010. Augmenting the Solow-Romer Model with Spatial Externalities: An Application to the Brazilian Case. *Encontro Brasileiro de Econometria*, **32**.

Krugman, Paul. 1991. Increasing Returns and Economic Geography. *Journal of Political Economy*, **99**(3), 483–99.

Krugman, Paul, & Venables, Anthony J. 1995 (Apr.). *Globalization and the Inequality of Nations*. NBER Working Papers 5098. National Bureau of Economic Research, Inc.

Ramsey, Frank P. 1928. A Mathematical Theory of Saving. *The Economic Journal*, **38**(152), 543 – 559.

Romer, Paul M. 1986. Increasing returns and long run growth. *Journal of Political Economy*, **94**, 1002 1037.

Solow, Robert M. 1956. A Contribution to the Theory of Economic Growth. *The Quarterly Journal of Economics*, **70**(1), 65 – 94.